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OSCILLATIONS GENERATED IN THE COMPRESSION OF A VISCOELASTIC BODY

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Studies of the various kinds of instability that evolve in processes of deformation of viscoelastic materials have been reported in a large number of papers (see [1] and [2-7]). Mainly elongational flows of incompressible liquids have been investigated in these studies. As a rule, the inertial terms in the equation of motion are neglected in the analysis of stability.

It is shown below that in deformations of compressible viscoelastic bodies another type of instability is possible, namely oscillations induced by the bulk elasticity of the material and driven up in hydrostatic compression of the sample as a result of inertial interaction of the disturbances with the main flow. The conditions for growth of the oscillations are determined from the linearized small-perturbation equations; on the basis of a nonlinear analysis, the nature of the excitation is investigated, and the amplitude of the oscillations is determined.

Calculations are carried out for the case of uniform planar deformation and ideally smooth and rigid bounding surfaces. These assumptions clearly do not play a vital role in the investigated instability mechanism.

1. Let us consider a rectangular compressible viscoelastic body (Fig. 1) bounded by smooth rigid planes. The planes $x_1 = \pm L(t)$ move symmetrically relative to one another with a velocity $U = dL/dt$, and the planes $x_2 = \pm R(t)$ do likewise with a velocity $V = dR/dt$. The velocities U and V can be either positive (extension) or negative (compression) and are assumed to be constant, so that the dimensions of the sample vary with time according to the linear law

$$L = L_0 + Ut,\ R = R_0 + Vt. \tag{1.1}$$

The equations describing the isothermal flow of the sample material have the form (repeated indices signify summation)

$$\partial\rho/\partial t + \partial(\rho v_k)/\partial x_k = 0; \tag{1.2}$$

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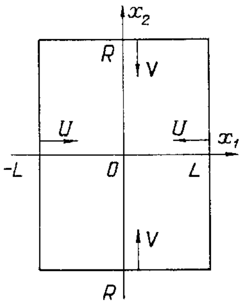

Fig. 1

$$\rho(\partial v_i/\partial t + v_k \partial v_i/\partial x_k) = \partial \sigma_{ki}/\partial x_k, \tag{1.3}$$

where ρ is the density, v_i denotes the components of the velocity, and σ_{ik} are the components of the stress tensor.

The rheological equation of the material is assumed to have the form

$$\sigma_{ij} = K e_{kk} \delta_{ij} + \tau_{ij}, \tag{1.4}$$

where K is the bulk modulus and e_{ij} are the components of the Almansi strain tensor [8], which are related to v_i by the equations

$$\partial e_{ij}/\partial t + v_k \partial e_{ij}/\partial x_k + e_{kj} \partial v_k/\partial x_i + e_{ik} \partial v_k/\partial x_j = (1/2)(\partial v_i/\partial x_j + \partial v_j/\partial x_i). \tag{1.5}$$

The deviatoric stresses τ_{ij} obey the equation (Maxwell model) [9]

$$\tau_{ij} + \lambda \delta \tau_{ij}/\delta t = 2\eta \xi_{ij}. \tag{1.6}$$

Here ξ_{ij} is the deviatoric part of the strain-rate tensor:

$$\xi_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) - \frac{1}{3}\frac{\partial v_k}{\partial x_k}\delta_{ij}; \tag{1.7}$$

η is the shear viscosity, λ is the relaxation time, and the symbol $\delta/\delta t$ denotes the Oldroyd convected derivative. The calculations are carried out for the two cases in which (1.6) incorporates either the upper convectional derivative

$$\frac{\delta \tau_{ij}}{\delta t} = \frac{\partial \tau_{ij}}{\partial t} + v_k \frac{\partial \tau_{ij}}{\partial x_k} - \frac{\partial v_i}{\partial x_k}\tau_{kj} - \frac{\partial v_j}{\partial x_k}\tau_{ik}, \tag{1.8}$$

or the lower convectional derivative

$$\frac{\delta \tau_{ij}}{\delta t} = \frac{\partial \tau_{ij}}{\partial t} + v_k \frac{\partial \tau_{ij}}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\tau_{kj} + \frac{\partial v_k}{\partial x_j}\tau_{ik}. \tag{1.9}$$

Hereinafter all intermediate calculations refer to the case (1.8), but the final results are given for both cases.

For planar deformation, relations (1.2)-(1.8) are reducible to a system of nine equations for the functions ρ, v_1, v_2, τ_{11}, τ_{22}, τ_{12}, e_{11}, e_{22}, e_{12}. These functions must satisfy the boundary conditions

$$\begin{gathered} v_1(L, x_2, t) = U, \; v_2(x_1, R, t) = V, \\ \tau_{12}(L, x_2, t) = 0, \; \tau_{12}(x_1, R, t) = 0 \end{gathered} \tag{1.10}$$

and the symmetry conditions

$$\begin{gathered} v_1(0, x_2, t) = 0, \; v_2(x_1, 0, t) = 0, \\ v_1(x_1, x_2, t) = -v_1(-x_1, x_2, t), \; v_1(x_1, x_2, t) = v_1(x_1, -x_2, t), \\ v_2(x_1, x_2, t) = v_2(-x_1, x_2, t), \; v_2(x_1, x_2, t) = -v_2(x_1, -x_2, t). \end{gathered} \tag{1.11}$$

Equations (1.2)-(1.8) subject to conditions (1.10) and (1.11) admit a solution corresponding to uniform deformation of the sample:

$$\begin{gathered} \rho^0 = \frac{C}{L(t)R(t)}, \quad v_1^0 = U\frac{x_1}{L(t)}, \quad v_2^0 = V\frac{x_2}{R(t)}, \quad \tau_{ij}^0 = \tau_{ij}^0(t), \\ e_{ij}^0 = e_{ij}^0(t), \end{gathered} \tag{1.12}$$

where $C = \rho_0 L_0 R_0$ (ρ_0 is the density at the initial time). The quantities τ_{ij}^0 and e_{ij}^0 are given by the conditions

$$\frac{de_{11}^0}{dt} + 2\frac{U}{L}e_{11}^0 = \frac{U}{L}, \quad \frac{de_{22}^0}{dt} + 2\frac{V}{R}e_{22}^0 = \frac{V}{R}, \quad e_{12}^0 = 0,$$
$$\lambda\frac{d\tau_{11}^0}{dt} + \left(1 - 2\lambda\frac{U}{L}\right)\tau_{11}^0 = \frac{2}{3}\eta\left(2\frac{U}{L} - \frac{V}{R}\right), \quad \lambda\frac{d\tau_{22}^0}{dt} + \left(1 - 2\lambda\frac{V}{R}\right)\tau_{22}^0 = \frac{2}{3}\eta\left(2\frac{V}{R} - \frac{U}{L}\right), \quad \tau_{12}^0 = 0. \tag{1.13}$$

2. We now analyze the stability of the uniform deformation (1.12) under small disturbances. It will be helpful to transform to the new set of variables (x, y, ζ):

$$x = x_1/L(t), \ y = x_2/R(t), \ \zeta = t,$$

where ζ has the significance of explicitly occurring time. The derivatives with respect to x_1, x_2, and t in (1.2)-(1.8) are expressed in terms of the new variables as follows:

$$\frac{\partial}{\partial x_1} = \frac{1}{L}\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial x_2} = \frac{1}{R}\frac{\partial}{\partial y}, \ \frac{\partial}{\partial t} = \frac{\partial}{\partial \zeta} - \frac{U}{L}x\frac{\partial}{\partial x} - \frac{V}{R}y\frac{\partial}{\partial y}.$$

It is also convenient, in place of ρ, to use the variable $r = \rho LR$, which has a constant value for the uniform deformation (1.12).

We write the solution of the system (1.2)-(1.8) in the form

$$r = C + \Phi, \quad v_1 = Ux + u_1, \quad v_2 = Vy + u_2,$$
$$\tau_{ij} = \tau_{ij}^0 + T_{ij}, \quad e_{ij} = e_{ij}^0 + E_{ij}. \tag{2.1}$$

After linearization with respect to the small disturbances we arrive at the equations

$$\frac{\partial \Phi}{\partial \zeta} + C\left(\frac{1}{L}\frac{\partial u_1}{\partial x} + \frac{1}{R}\frac{\partial u_2}{\partial y}\right) = 0; \tag{2.2}$$

$$C\left(\frac{\partial u_1}{\partial \zeta} + \frac{U}{L}u_1\right) = KR\frac{\partial}{\partial x}(E_{11} + E_{22}) + R\frac{\partial T_{11}}{\partial x} + L\frac{\partial T_{12}}{\partial y}; \tag{2.3}$$

$$C\left(\frac{\partial u_2}{\partial \zeta} + \frac{V}{R}u_2\right) = KL\frac{\partial}{\partial y}(E_{11} + E_{22}) + L\frac{\partial T_{22}}{\partial y} + R\frac{\partial T_{12}}{\partial x}; \tag{2.4}$$

$$T_{11} + \lambda\left\{\frac{\partial T_{11}}{\partial \zeta} - \frac{2}{L}\tau_{11}^0\frac{\partial u_1}{\partial x} - 2\frac{U}{L}T_{11}\right\} = \frac{2}{3}\eta\left(\frac{2}{L}\frac{\partial u_1}{\partial x} - \frac{1}{R}\frac{\partial u_2}{\partial y}\right); \tag{2.5}$$

$$T_{22} + \lambda\left\{\frac{\partial T_{22}}{\partial \zeta} - \frac{2}{R}\tau_{22}^0\frac{\partial u_2}{\partial y} - 2\frac{V}{R}T_{22}\right\} = \frac{2}{3}\eta\left(\frac{2}{R}\frac{\partial u_2}{\partial y} - \frac{1}{L}\frac{\partial u_1}{\partial x}\right); \tag{2.6}$$

$$T_{12} + \lambda\left\{\frac{\partial T_{12}}{\partial \zeta} - \frac{1}{L}\tau_{11}^0\frac{\partial u_2}{\partial x} - \frac{1}{R}\tau_{22}^0\frac{\partial u_1}{\partial y} - \left(\frac{U}{L} + \frac{V}{R}\right)T_{12}\right\} = \eta\left(\frac{1}{R}\frac{\partial u_1}{\partial y} + \frac{1}{L}\frac{\partial u_2}{\partial x}\right); \tag{2.7}$$

$$\frac{\partial E_{11}}{\partial \zeta} + \frac{2}{L}e_{11}^0\frac{\partial u_1}{\partial x} + 2\frac{U}{L}E_{11} = \frac{1}{L}\frac{\partial u_1}{\partial x}; \tag{2.8}$$

$$\frac{\partial E_{22}}{\partial \zeta} + \frac{2}{R}e_{22}^0\frac{\partial u_2}{\partial y} + 2\frac{V}{R}E_{22} = \frac{1}{R}\frac{\partial u_2}{\partial y}; \tag{2.9}$$

$$\frac{\partial E_{12}}{\partial \zeta} + \frac{1}{L}e_{11}^0\frac{\partial u_2}{\partial x} + \frac{1}{R}e_{22}^0\frac{\partial u_1}{\partial y} + \left(\frac{U}{L} + \frac{V}{R}\right)E_{12} = \frac{1}{2}\left(\frac{1}{L}\frac{\partial u_1}{\partial y} + \frac{1}{R}\frac{\partial u_2}{\partial x}\right). \tag{2.10}$$

We obtain the boundary conditions for Eqs. (2.2)-(2.10) by substituting expressions (2.1) into (1.10); they correspond to zero normal components of the disturbances of the velocity and the tangential stresses. The velocities along the boundary and the normal stresses at the boundaries can be arbitrary in this case, growing with the development of instability.

The solution of the system (2.2)-(2.10) subject to the boundary conditions and symmetry conditions has the form

$$u_1 = \alpha_1(\zeta)\sin \pi n x \cdot \cos \pi m y, \ u_2 = \alpha_2(\zeta)\cos \pi n x \cdot \sin \pi m y,$$
$$\{\Phi, T_{11}, T_{22}, E_{11}, E_{22}\} = \{F(\zeta), S_1(\zeta), S_2(\zeta), \beta_1(\zeta), \beta_2(\zeta)\}\cos \pi n x \cdot \cos \pi m y,$$
$$\{T_{12}, E_{12}\} = \{S_3(\zeta), \beta_3(\zeta)\}\sin \pi n x \cdot \sin \pi m y. \tag{2.11}$$

The substitution of expression (2.11) into (2.2)-(2.10) yields a system of ordinary differential equations for the amplitudes of the disturbances α_1, α_2, β_1, β_2, S_1, S_2, S_3 [Eqs. (2.2) and (2.10) are separable for the quantities F and β_3]. The analysis of the solutions of this system is simplified if it is confined to similarity-preserving deformations: $U/V = L_0/R_0$ $(U/L = V/R)$. Then, as is evident from (1.13),

$$e_{11}^0 = e_{22}^0 = e^0, \quad \tau_{11}^0 = \tau_{22}^0 = \tau^0, \quad \frac{de^0}{dt} + 2\frac{U}{L}e^0 = \frac{U}{L},$$

$$\lambda \frac{d\tau^0}{dt} + \left(1 - 2\lambda\frac{U}{L}\right)\tau^0 = \frac{2}{3}\eta\frac{U}{L}.$$

The solutions of these equations are written as follows after transformation to the new independent variable $G = L/L_0$:

$$e^0 = \frac{1}{2}\left(1 - \frac{1}{G^2}\right),$$

$$\tau^0 = \frac{\eta}{3\lambda}\left\{A^2G^2e^{-AG}\left[\ln G + \sum_{n=1}^{\infty}\frac{A^n(G^n - 1)}{n\cdot n!}\right] + G^2(A+1)e^{A(1-G)} - (1 + AG)\right\}, \tag{2.12}$$

$$A = \frac{L_0}{\lambda U}.$$

The equations for the amplitudes of the disturbances for $U/L = V/R$ can be reduced to a system of three equations for the quantities

$$\alpha = \pi n\alpha_1/U + \pi m\alpha_2/V, \ \beta = \beta_1 + \beta_2,$$
$$S = \pi^2n^2(V/U)S_1 + \pi^2m^2(U/V)S_2 - 2\pi^2mnS_3,$$

which, after the substitution in explicit form of the expression for e^0, is written in the form

$$C\left(\frac{d\alpha}{d\zeta} + \frac{U}{L}\alpha\right) = -\frac{L}{U}(Kl^2\beta + S); \tag{2.13}$$

$$\frac{d\beta}{d\zeta} + 2\frac{U}{L}\beta = \frac{UL_0^2}{L^3}\alpha; \tag{2.14}$$

$$S + \lambda\left(\frac{dS}{d\zeta} - 2\frac{U}{L}S\right) = 2\left(\lambda\tau^0 + \frac{2}{3}\eta\right)\frac{U}{L}l^2\alpha, \tag{2.15}$$

where

$$l^2 = \pi^2n^2(V/U) + \pi^2m^2(U/V).$$

The numerical solution of Eqs. (2.13)-(2.15) shows that under definite conditions the time variation of the disturbances has an oscillatory behavior. An analytical investigation of the conditions for the onset of oscillations is readily carried out on the assumption that the period of the oscillations is much smaller than the characteristic rise time of the main flow $L_0/|U|$. Then, in accordance with the notions of the method of two-scale decompositions we introduce the "slow" time $t' = t|U|/L_0$ and the "fast" time $\vartheta = \Omega t$ (Ω is the frequency of the oscillations, $\Omega \gg |U|/L_0$), writing the time derivative as the sum of the partial derivatives with respect to the variables t' and ϑ:

$$\frac{\partial}{\partial t} = \Omega\frac{\partial}{\partial\vartheta} + \frac{|U|}{L_0}\frac{\partial}{\partial t'}. \tag{2.16}$$

The amplitudes of the disturbances α, β, S are, by assumption, functions of the fast time only ($\partial/\partial\zeta = \Omega\partial/\partial\vartheta$), whereas according to (1.1) and (2.12) the quantities L, R, τ^0 depend only on the slow time [$L/L_0 = 1 \pm t'$, $R/R_0 = 1 \pm (L_0/R_0)t'$, $\tau^0 = \tau^0(L/L_0)$]. Consequently, L, R, τ^0 can be regarded as constants in relation to differentiation with respect to ζ (or ϑ), and the system (2.13)-(2.15) is reducible to a single equation for α (the equations for β and S have the same form):

$$\lambda\frac{d^3\alpha}{d\zeta^3} + \left(1 + \lambda\frac{U}{L}\right)\frac{d^2\alpha}{d\zeta^2} + \left[\frac{2Hl^2}{C} + \frac{3U}{L} + \lambda\left(\frac{Kl^2L_0^2}{CL^2} - \frac{4U^2}{L^2}\right)\right]\frac{d\alpha}{d\zeta} +$$
$$+ \left[\frac{Kl^2L_0^2}{CL^2} + \frac{4HUl^2}{CL} + \frac{2U^2}{L^2} - \lambda\frac{2U}{L}\left(\frac{Kl^2L_0^2}{CL^2} + \frac{2U^2}{L^2}\right)\right]\alpha = 0, \tag{2.17}$$

where we have introduced the notation $H = \lambda\tau^0 + 2\eta/3$.

We first consider the case $\lambda = 0$, when elasticity effects occur only in connection with bulk deformations of the material and the equation for α acquires the form

$$\frac{d^2\alpha}{d\zeta^2} + \left(\frac{4\eta l^2}{3C} + \frac{3U}{L}\right)\frac{d\alpha}{d\zeta} + \left(\frac{Kl^2L_0^2}{CL^2} + \frac{8\eta Ul^2}{3CL} + \frac{2U^2}{L^2}\right)\alpha = 0. \tag{2.18}$$

The solution of Eq. (2.18) can be written in the form

$$\alpha = ae^{-\delta\zeta}\cos(\Omega\zeta + \varphi),$$

where

$$\delta = 2\eta l^2/3C + 3U/2L,$$

$$\Omega^2 = \frac{Kl^2L_0^2}{CL^2} + \frac{2U^2}{L^2} + \frac{8\eta Ul^2}{3CL} - \frac{1}{4}\left(\frac{4\eta l^2}{3C} + \frac{3U}{L}\right)^2. \tag{2.19}$$

For $\Omega^2 > 0$ the time variation of the disturbances has an oscillatory behavior. In the case of extension of the sample ($U > 0$) the oscillations are always damped ($\delta > 0$), whereas in compression ($U < 0$) they can grow. The oscillatory growth of the disturbances ($\delta < 0$) begins at the time when L falls below a certain critical value

$$L_* = 9|U|C/4\eta l^2.$$

It is evident from (2.19) that the assumption of a rapid time variation of the disturbances ($\Omega \gg |U|/L_0$), used in the derivation of (2.17) and (2.18), is valid for fairly large values of K. Assuming that the characteristic viscous-damping time C/η is of the same order of magnitude as the characteristic time of the main flow $L_0/|U|$, under the condition

$$K \gg CU^2/L_0^2 \tag{2.20}$$

we can represent the expression (2.19) for the frequency of the oscillations in the form

$$\Omega = (KL_0^2l^2/CL^2)^{1/2}[1 + O(\psi)], \tag{2.21}$$

where the quantity

$$\psi = CU^2/KL_0^2 \tag{2.22}$$

characterizes the order of smallness of the discarded terms [by (2.20) we have $\psi \ll 1$]. The ratio of the oscillation period $2\pi/\Omega$ to the characteristic time of the main flow for large values of K is of the same order of smallness ψ (fast oscillations).

Going to the general case $\lambda \neq 0$, we investigate the solutions of Eq. (2.17) in the large-K limit ($\psi \ll 1$). We first determine the roots h_i of the characteristic polynomial, assuming that the characteristic viscous-damping time, the relaxation time, and the characteristic time of the main flow are of the same order. Then the characteristic polynomial corresponding to (2.17) can be written

$$\left(h\frac{L}{U}\right)^3 + a_1\left(h\frac{L}{U}\right)^2 + a_2[1 + O(\psi)]\left(h\frac{L}{U}\right) + a_3[1 + O(\psi)] = 0, \tag{2.23}$$

where

$$a_1 = \left(1 + \frac{L}{U\lambda}\right);\ a_2 = \frac{l^2}{C}\frac{KL_0^2}{U^2};\ a_3 = \frac{l^2}{C}\frac{KL_0^2}{U^2}\left(\frac{L}{U\lambda} - 2\right).$$

Noting that $a_1 \sim 1$, $a_2 \sim 1/\psi$, $a_3 \sim 1/\psi$, we write the expression for the pair of complex-conjugate roots of the polynomial in the form

$$h_{1,2} = \frac{U}{L}\left\{\left[\frac{a_3 - a_1a_2}{2a_2} + O(\psi)\right] \pm i\left[\sqrt{a_2} + O(\psi)\right]\right\}. \tag{2.24}$$

After substituting the expressions for a_1, a_2, a_3 we arrive at the equation (correct to terms of order ψ)

$$h_{1,2} = -3U/2L \pm i(Kl^2L_0^2/CL^2)^{1/2}.$$

Here the oscillation frequency is again described by (2.21), and the oscillations grow only in the case of compression of the sample for any values of L.

Threshold effects for growth of the oscillations can also be observed for $\lambda \neq 0$ if the viscosity η is large enough for the viscous terms in (2.17) to be the same order as the terms containing K. Under the condition (2.20) and for $\lambda \sim L_0/|U|$ [where $\tau^0 \sim \eta/\lambda$, see (2.12)] it is necessary that

$$(C/\eta)/(L_0/|U|) \sim \psi \tag{2.25}$$

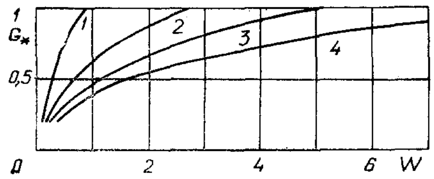

Fig. 2

(the viscous-damping time is much smaller than the characteristic time of the flow). Under conditions (2.20) and (2.25) the characteristic polynomial again has for form (2.23), where

$$a_1 = 1 + \frac{L}{U\lambda}, \quad a_2 = \frac{l^2}{C}\left(\frac{KL_0^2}{U^2} + \frac{2HL^2}{\lambda U^2}\right),$$
$$a_3 = \frac{l^2}{C}\left[\frac{KL_0^2}{U^2}\left(\frac{L}{UL} - 2\right) + \frac{4HL^2}{\lambda U^2}\right].$$

From expression (2.24), after certain transformations, we obtain an expression for the pair of complex-conjugate roots:

$$h_{1,2} = \frac{-3\lambda KUL_0^2 + 2HL^3(U/L - 1/\lambda)}{2\lambda KL_0^2 L + 4HL^3} \pm i\left(\frac{Kl^2L_0^2}{CL^2} + \frac{2Hl^2}{C\lambda}\right)^{1/2}. \tag{2.26}$$

The growth (decay) rate in (2.26) can change sign, i.e., for large viscosities, as in the case $\lambda = 0$, the oscillations can grow only after compression to some critical length L_*. An expression for L_*, which is determined from the condition that the real part of (2.26) vanishes, cannot be written out explicitly, because L enters into (2.26) in a complex way through τ^0 [see (2.12)]. The corresponding transcendental equation is solved numerically by the method of chords.

In the ensuing discussion it is useful to transform to dimensionless quantities. We use the dimensionless parameters

$$W = \lambda|U|/L_0, \; B = \eta/\rho_0 L_0|U|, \; D = K/\rho_0 U^2 \tag{2.27}$$

and the quantities

$$G = L/L_0, \; \omega = \Omega L_0/|U|, \; \Theta = \tau^0 L_0/\eta|U|. \tag{2.28}$$

The equation governing the threshold length L_* (or G_*) and the expression for the frequency of the oscillations at the threshold are written in the following form for $U < 0$ [these expressions are obtained from the dimensionless equation (2.26) when the real part of h is equal to zero):

$$3D - 2BG_*^2(\Theta_* + 2/3W)(1 + G_*/W) = 0, \tag{2.29}$$
$$\omega_*^2 = l^2\left[D/G_*^2 + 2B(\Theta_* + 2/3W)\right].$$

Figure 2 shows the threshold length G_* as a function of the dimensionless relaxation time W (Weissenberg number) for U = V and n = m = 1 for B/D = 1, 5, 10, 20 (curves 1-4 respectively). It is seen that as the relaxation time is increased the value of G_* approaches unity ($L_* \to L_0$), so that for large W the oscillations are driven up for all $L \leqslant L_0$. Increasing the viscosity B, as expected, pulls back the threshold for the inception of growing oscillations.

We also give an expression for the pair of complex-conjugate roots of the characteristic polynomial [analogous to (2.26)] in the case where the lower convectional derivative (1.9) is used in (1.6):

$$h_{1,2} = \frac{-3\lambda KUL_0^2 - 2HL^3(3U/L + 1/\lambda)}{2\lambda KL_0^2 L + 4HL^3} \pm i\left(\frac{Kl^2L_0^2}{CL^2} + \frac{2Hl^2}{C\lambda}\right)^{1/2}.$$

Here $H = -\lambda\tau^0 + 2\eta/3$, and τ^0 is calculated from the expression

$$\tau^0 = -\frac{2\eta}{3\lambda}\frac{1}{G^2A^2}\left[(A-1)e^{A(1-G)} - (AG-1)\right], \quad A = \frac{L_0}{\lambda U}. \tag{2.30}$$

The foregoing analysis of the conditions for the generation of oscillations was based on Eq. (2.17), which is valid for describing the behavior of the disturbances only at high

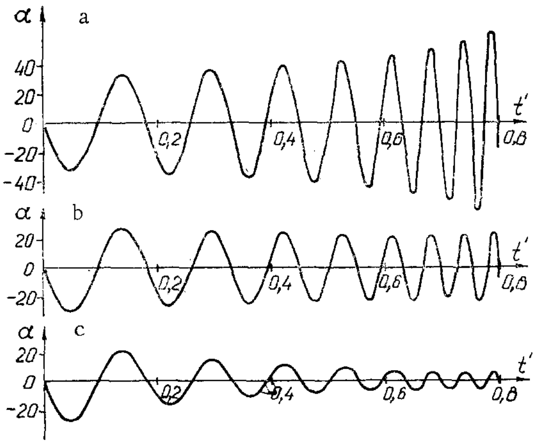

Fig. 3

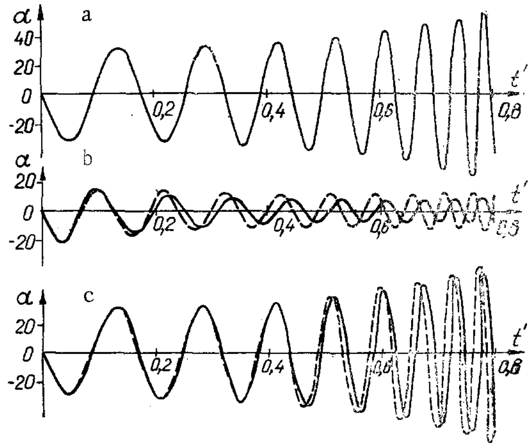

Fig. 4

oscillation frequencies, which correspond to large values of D and B (K and η). We have investigated the behavior of the disturbances for arbitrary values of the parameters by integrating Eqs. (2.13)-(2.15) numerically according to the fourth-order Runge–Kutta method.

The results of the numerical solution for large values of D and B corroborate the laws described above. For values of the parameters such that the period of the oscillations is commensurate with the characteristic rise time of the main flow, the numerical solution shows that the qualitative conclusions pertaining to large D and B remain valid in this situation as well. As an example, Figs. 3 and 4 show the waveform of the oscillations (α as a function of the dimensionless time $t' = t|U|/L_0$) for certain values of the parameters. All the curves in Figs. 3 and 4 are plotted for U = V, n = m = 1, D = 50, and identical initial conditions (α = S = 0, β = 1). The dashed curves represent the functions α(t') obtained using the lower convectional derivative (1.9) in (1.6).

The curves in Figs. 3a-c illustrate how the nature of the oscillations depends on the viscosity B = 0.01, 0.1, 0.25 respectively for W = 0. The influence of W on the waveform is discerned from a comparison of Figs. 3c and 4a, and of Figs. 4b and c. As for $\omega \gg 1$ (see Fig. 2), an increase in the relaxation time is a destabilizing factor. In all the curves there is a noticeable increase in the frequency of the oscillations with the passage of time (with decreasing L), as predicted by (2.21) and (2.29). In Figs. 4a-c, respectively, W = 0.1, 0.1, 1 and B = 0.25, 5, 5.

3. The finite-amplitude oscillations established after the threshold, $L < L_*$, like the solution of the problem of the nature of the excitation of oscillations at $L = L_*$, must be investigated on the basis of the complete nonlinear equations (1.2)-(1.8). It is natural to consider the solutions of these equations in the high-frequency limit, because the notion of steady-state oscillatory states presupposes smallness of the oscillation period relative to the characteristic time of the main flow. This limit is reached under the conditions

$$D \gg 1,\ B \gg 1, \tag{3.1}$$

which are equivalent to (2.20), (2.22), and (2.25).

Equations (1.2)-(1.8) are written in dimensionless variables measured relative to the corresponding quantities for the main flow

$$u_1' = \frac{v_1 - v_1^0}{|U|},\quad u_2' = \frac{v_2 - v_2^0}{|V|},\quad \Phi' = \frac{r - C}{C},\quad T'_{ij} = \frac{\tau_{ij} - \tau_{ij}^0}{\eta |U|/L_0}, \qquad E'_{ij} = e_{ij} - e_{ij}^0. \tag{3.2}$$

For values of L close to the threshold, solutions of the nonlinear equations are sought in the form of series in a small parameter ε having the significance of the steady-state oscillation amplitude:

$$\Phi' = \varepsilon\Phi^{(1)} + \varepsilon^2\Phi^{(2)} + \ldots,\quad u_i' = \varepsilon u_i^{(1)} + \varepsilon^2 u_i^{(2)} + \ldots, \qquad T'_{ij} = \varepsilon T_{ij}^{(1)} + \varepsilon^2 T_{ij}^{(2)} + \ldots,\quad E'_{ij} = \varepsilon E_{ij}^{(1)} + \varepsilon^2 E_{ij}^{(2)} + \ldots \tag{3.3}$$

Besides (3.3), we write the formal expansion

$$G = G_* + \varepsilon G_1 + \varepsilon^2 G_2 + \ldots, \tag{3.4}$$

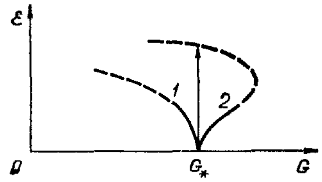

Fig. 5

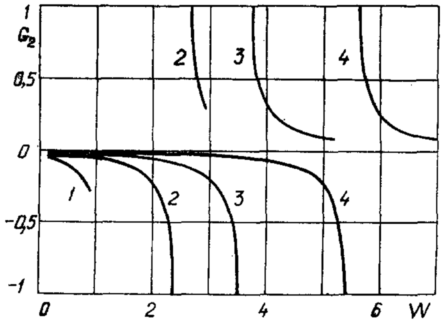

Fig. 6

which determines ε, and the expansion for ω

$$\omega = \omega_* + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \ldots, \tag{3.5}$$

which determines the nonlinear frequency shift. The quantity Θ [the dimensionless version of τ^0, see (2.28)] in the equations is expressed in terms of G by means of (2.12) and is therefore also represented by a series in ε:

$$\Theta = \Theta_* + \varepsilon\Theta_1 + \varepsilon^2\Theta_2 + \ldots,$$
$$\Theta_1 = \left(\frac{d\Theta}{d\varepsilon}\right)_{G=G_*} = \left(\frac{d\Theta}{dG}\right)_* G_1, \quad \Theta_2 = \frac{1}{2}\left(\frac{d^2\Theta}{dG^2}\right)_* G_1^2 + \left(\frac{d\Theta}{dG}\right)_* G_2. \tag{3.6}$$

Substituting the expressions (3.3)-(3.6) into the dimensionless equations (1.2)-(1.8) in each order with respect to ε, we obtain a system of linear differential equations, which can be written in the form

$$N\mathbf{Z}^{(k)} = \mathbf{f}^{(k)}, \tag{3.7}$$

where $\mathbf{Z}^{(k)}$ is a column vector formed from the quantities $\Phi^{(k)}$, $u_i^{(k)}$, $T_{ij}^{(k)}$, $E_{ij}^{(k)}$; N is the matrix differential operator corresponding to the dimensionless homogeneous system (2.2)-(2.10) for $G = G_*$ and $\omega = \omega_*$. The right-hand sides $\mathbf{f}^{(k)}$ are expressed in terms of the quantities computed in the preceding orders with respect to ε.

The system (3.7) is homogeneous in the first order with respect to ε; the condition of periodicity of its solution yields relations (2.29) governing the quantities G_* and ω_*. The solution of the homogeneous system has the form (2.11); to simplify the calculations in the higher orders with respect to ε we set $L_0 = R_0$, $U = V$, and $n = m$.

In the higher orders with respect to ε the right-hand sides of (3.7) are not zero, and a periodic solution of the inhomogeneous system exists only under the condition that $\mathbf{f}^{(k)}$ is orthogonal to the solution of the associated homogeneous system. This condition determines the constants G_k and ω_k in the expansions (3.4) and (3.5). The second-approximation equations yield

$$G_1 = \omega_1 = 0; \tag{3.8}$$

from the third-approximation equation we determine the quantity

$$G_2 = \frac{\pi^2 n^2\left[2G_*^2 B\left(\Theta_* + \frac{2}{3W}\right) - 3D\right]\left[G_*^2 B\left(\Theta_* + \frac{2}{3W}\right) - 3D\right]\left(1 + \frac{G_*}{W}\right)}{24\left(\frac{3}{W} + \frac{4}{G_*}\right)\left(4 + \frac{G_*}{W}\right)\left[G_*^3 B\left(\frac{d\Theta}{dG}\right)_* - D\right]} \tag{3.9}$$

[the correction ω_2 is absent in the limit (3.1)].

Restricting the expansion (3.4) to second-order terms in ε, on the basis of (3.8) and (3.9) we find the dependence of the steady-state amplitude ε on the instantaneous length of the sample:

$$\varepsilon = \left(\frac{G - G_*}{G_2}\right)^{1/2} = \left(\frac{L - L_*}{L_0 G_2}\right)^{1/2}. \tag{3.10}$$

An analysis of expression (3.10) shows that the nature of the excitation of the oscillations upon attainment of the critical length L_* depends on the sign of G_2. For $G_2 < 0$ the quantity ε is real-valued in the domain $L < L_*$, i.e., the amplitude of the oscillations increases, beginning with $L = L_*$, in continuous fashion [soft excitation, corresponding to the function ε(G), is represented qualitatively by curve 1 in Fig. 5]. In the opposite case $G_2 > 0$, hard excitation takes place, and at $L = L_*$ the amplitude of the oscillations jumps abruptly to a "finite" value [curve 2 in Fig. 5, the solid part corresponding to expression (3.10)].

The quantity G_2 determined from (3.9) is shown in Fig. 6 as a function of the Weissenberg number W for B/D = 1, 5, 10, 20 (curves 1-4 respectively; the corresponding values of G_* can be found in Fig. 2). It is seen that for small relaxation times soft excitation always takes place ($G_2 < 0$), but with an increase in W the quantity G_2 can change sign.

Next we give the expression obtained for G_2 using the lower convectional derivative (1.9) in (1.6):

$$G_2 = -\frac{3\pi^2 n^2 \left[2G_*^2 B\left(\frac{2}{3W} - \Theta_*\right) + D\right]}{16\left[G_*^3 B\left(\frac{d\Theta}{dG}\right)_* + D\right]\left(3\frac{G_*}{W} - 8\right)}$$

[here the quantity Θ is calculated in terms of the τ^0 given by expression (2.30)]. An analysis of this expression shows that G_2 is always negative, i.e., in this case only soft excitation takes place.

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